Indian Statistical Institute
M.Math. I Year
First Semester 2006-07
Mid Semester Examination
Advanced Probability

Time: 3 hrs | Date:18-09-06 | Max. Marks: 100

1. (a) Suppose F_1 and F_2 are two right continuous non decreasing functions on the real line and μ a measure such that $\mu(a, b] = F_1(b) - F_1(a) = F_2(b) - F_2(a)$. Show that there exists α such that $F_1(x) = F_2(x) + \alpha$. (4)

   (b) If $\mu(\mathbb{R}) < \infty$ show that there exists a unique non decreasing right continuous function F such that $\lim_{x\to-\infty} F(x) = 0; \lim_{x\to\infty} F(x) = \mu(\mathbb{R})$ and $\mu(a, b] = F(b) - F(a)$. (4)

   (c) Let μ be a finite measure on the Borel sigma field of a topological space X. Show that $\mu(A) = \sup\{\mu(C) : C \subset A, C \text{ closed}\}$ iff $\mu(A) = \inf\{\mu(D) : A \subset D, D \text{ open}\}$. (3)

   (d) Give an example of an infinite measure which is not regular. (5)

2. (a) Let X_1 and X_2 be independent standard normal random variables. Let $\Theta = \tan^{-1}(\frac{X_2}{X_1})$ and $R = \sqrt{X_1^2 + X_2^2}$. Calculate the joint distribution of (Θ, R). Show that Θ and R are independent. (5+3)

   (b) Show that the characteristic function of a standard normal random variable is given by the function $\exp(-\frac{t^2}{2})$. (8)

3. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $X_n \to X$ in probability. show that $f(X_n) \to f(X)$ in probability. (6)

   (b) Suppose that $X_n \Longrightarrow X$ and $\delta_n \to 0$. Show that $\delta_n X_n \Longrightarrow 0$. (7)

   (c) . If μ_n is the uniform distribution on $[-n, n]$ then show that μ_n does not converge weakly to μ , for any μ. (5)

4. (a) Let (X_n) be a sequence of i.i.d. random variables . Let $S_0 = 0, S_n = X_1 + \cdots + X_n$. Let $W_0 = 0$ and $W_n = \max\{0, W_n + X_n\}$. Show that $W_n = \max_{0\le k\le n}(S_n - S_k)$ (8)

   (b) Show that W_n has the same distribution as $\max_{0\le k\le n} S_k$. (8)

5. Let (X_n) be an i.i.d sequence , with $E|X_1| = \infty$.

(a) Show that $\sum_n P[|X_n| \geq an] = \infty$ for every a. (6)

(b) Show that $\sup_n |X_n| n^{-1} = \infty$ a.s (5)

(c) Show that $\sup_n |S_n| n^{-1} = \infty$ a.s (5)

6. (a) Let ϕ be the characteristic function of the measure μ. Show that for every a,

$$\mu\{a\} = lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \phi(t)\ dt$$

(10)

(b) Let (x_i) be the points of positive μ measure. Show that

$$lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)|^2\ dt = \sum_i (\mu(x_i))^2.$$

Hint : If X and Y are independent and identically distributed with distribution μ , calculate P(X - Y = 0) using a). (15)